AD-A166 562 MATCHING EXTENSION AND THE GENUS OF A GRAPH(U) 1/1
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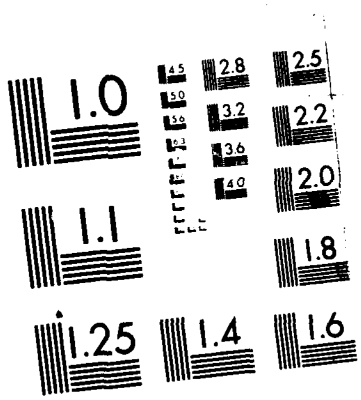

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Matching Extension and the Genus of a Graph

by

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Contract No. N00014-85-K-0488
Date: March 1986

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ABSTRACT

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Let G be a graph with p points having a perfect matching and suppose n is a positive integer with $n \leq (p-2)/2$. Then G is n-extendable if every matching in G containing n lines is a subset of a perfect matching. In this paper we obtain an upper bound on the n-extendability of a graph in terms of its genus. Keywords: [illegible]; Theorems

LIST OF SYMBOLS USED IN THIS PAPER:

1. γ Greek lower case gamma
2. Φ Greek upper case phi
3. $\lfloor$ the so-called "left floor" symbol
4. $\rfloor$ the so-called "right floor" symbol

NOTE: All Greek letters are circled in red in the submitted manuscript.

MATCHING EXTENSION AND THE GENUS OF A GRAPH

by

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1. Introduction and Terminology

Let G be a graph with $|V(G)| = p$ points and $|E(G)| = q$ lines. (Loops and parallel lines are forbidden in this paper.) A **matching** in G is any set of lines in $E(G)$ no two of which are adjacent. Matching M in G is said to be a **perfect matching**, or **p.m.**, if every point of G is covered by a line of M. Let G be any graph with a perfect matching and suppose positive integer $n \leq (p-2)/2$. Then G is **n-extendable** if every matching in G containing n lines is a subset of a p.m.

The concept of n-extendability gradually evolved from the study of elementary bipartite graphs (which are 1-extendable) (see Hetyei (1964), Lovász and Plummer (1977)), and then of arbitrary 1-extendable (or "matching-covered") graphs by Lovász (1983). The study of n-extendability for arbitrary n was begun by the author (1980).

The **genus** of graph G, $\gamma(G)$, is the minimum genus of all (orientable) surfaces in which G can be imbedded. Any imbedding of G in a surface of genus $\gamma = \gamma(G)$ is said to be a **minimal imbedding**. (For more information on the genus of a graph, see White (1973). In particular, recall the well-known result of Youngs (1963) which says that if graph G is imbedded in a surface of genus $\gamma = \gamma(G)$, then the (minimal) imbedding must be a 2-cell imbedding.)

A relationship between matching and genus was first studied by Nishizeki (1979) who treated the interplay between genus and the cardinality of a maximum matching.

In (1985), we showed that if G is planar, then G is not 3-extendable. Cook (1973) proved a result that implies that if G is a graph with genus $\gamma(G) = \gamma > 0$, then G is not $\lfloor \frac{1}{2}(5 + \sqrt{1 + 48\gamma}) \rfloor$ -extendable. In the present paper, we will improve on this result by showing that if $\gamma(G) > 0$, then G is not $\left(\left\lfloor \frac{9}{2} + \frac{18(\gamma-1)}{7+\sqrt{48\gamma-47}} \right\rfloor\right)$- extendable. Moreover, in the

* work supported by ONR Contract # N00014-85-K-0488

case of "toroidal" graphs (i.e., graphs of genus 1) we obtain structural information about – and an infinite family of – extremal graphs.

Throughout this paper, we will assume that all graphs are connected, that $\text{mindeg}(G) \geq 3$ and that $\text{mindeg}^*(G) \geq 3$, where $\text{mindeg}^*(G)$ denotes the size of a smallest face in an imbedding of G. We shall call a graph G **bicritical** if $G - u - v$ has a p.m. for every pair of points $u, v \in V(G)$. For any additional terminology, we refer the reader to Harary (1969), to Bondy and Murty (1976) or to Lovász and Plummer (1986).

2. The bound for arbitrary positive genus

One of our main tools will be the so-called *theory of Euler contributions* initiated by Lebesgue (1940) and further developed by Ore (1967) and by Ore and Plummer (1969). Let v be any point in a graph G minimally imbedded in a surface of genus $\gamma(G)$. Define the **Euler contribution** of v, $\Phi(v)$, by

$$\Phi(v) = 1 - \frac{\deg v}{2} + \sum_{i=1}^{\deg v} \frac{1}{x_i},$$

where the sum runs over the face angles at point v and x_i denotes the size of the ith face at v. (It is important to keep in mind that a face may contribute more than one face angle at a point v. Think of K_5 imbedded on the torus, for example.)

We next present several simple lemmas. We include the proofs for the sake of completeness. The first is essentially due to Lebesgue (1940).

2.1. LEMMA. *If a connected graph G is minimally imbedded in a surface of genus $\gamma = \gamma(G)$, then $\sum_v \Phi(v) = 2 - 2\gamma$.*

PROOF. Let $p = |V(G)|$, $q = |E(G)|$ and r be the number of faces in the imbedding. Then

$$\sum_v \Phi(v) = \sum_v \left(1 - \frac{\deg v}{2} + \sum_{i=1}^{\deg v} \frac{1}{x_i}\right) = p - q + r = 2 - 2\gamma,$$

by the generalized Euler formula. ∎

2.2. LEMMA. *Let G be minimally imbedded in a surface of genus $\gamma = \gamma(G)$ and have* $\operatorname{mindeg}^*(G) \geq 3$. *Then for all $v \in V(G)$, $\Phi(v) \leq 1 - \deg v/6$.*

PROOF. Since $x_i \geq 3$ for all i, we have $\Phi(v) \leq 1 - \deg v/2 + \deg v/3$ and the result follows. ■

2.3. LEMMA. *If G is connected and $p = |V(G)|$ and $\gamma = \gamma(G)$, then for some $v \in V(G)$, $\Phi(v) \geq (2-2\gamma)/p$.*

PROOF. The average value of $\Phi(v) = \frac{\sum \Phi(v)}{p} = \frac{(2-2\gamma)}{p}$ by Lemma 2.1, and the conclusion follows. ■

Let us agree to call any point $v \in V(G)$ which satisfies the inequality of Lemma 2.3 a **control point** (since such a point will be seen to "control" , or limit, the degree of matching extendability in G).

2.4. COROLLARY. *If G is connected and $\gamma = \gamma(G)$, then if v is any control point,*

$$\sum_{i=1}^{\deg v} \frac{1}{x_i} \geq \frac{(2-2\gamma)}{p} + \frac{\deg v}{2} - 1.$$

PROOF. Follows immediately from Lemma 2.3 and the definition of $\Phi(v)$. ■

2.5. LEMMA. *If G is connected and $\gamma = \gamma(G)$, then for any control point v,* $\deg v \leq 6 + 12(\gamma - 1)/p$.

PROOF. From Lemmas 2.2 and 2.3 we have $(2-2\gamma)/p \leq \Phi(v) \leq 1 - \deg v/6$ and the conclusion follows. ■

Next, we need a lower bound on the number of points in a graph of genus γ. The following is an immediate corollary of the Ringel-Youngs formula for the genus of the complete graph (1968).

2.6. LEMMA. *If G has p points and $\gamma(G) = \gamma > 0$, then $p > (7 + \sqrt{48\gamma - 47})/2$.*

PROOF. By the Ringel-Youngs result we have

$$\gamma = \gamma(G) \leq \gamma(K_p) = \left\lceil \frac{(p-3)(p-4)}{12} \right\rceil < \frac{(p-3)(p-4)}{12} + 1$$

and the inequality follows. ■

As our last preliminary results we state the next two theorems which will be used repeatedly below. The proofs may be found in Plummer (1980).

2.7. THEOREM. *If G is n-extendable for some $2 \leq n \leq (p-2)/2$, then G is also $(n-1)$-extendable.* ∎

2.8. THEOREM. *If G is n-extendable for some $1 \leq n \leq (p-2)/2$, then G is $(n+1)$-connected.* ∎

We are now prepared for the main result of this paper.

2.9. THEOREM. *If G is any connected graph with $\gamma(G) = \gamma > 0$, then*

(a) G is not $\left(\left\lfloor \frac{9}{2} + \frac{18(\gamma-1)}{7+\sqrt{48\gamma-47}} \right\rfloor\right)$ - extendable, while if (b) in addition, G contains no triangle, then G is not $(2 + \lfloor 2\sqrt{\gamma} \rfloor)$- extendable.

PROOF. If G contains no triangle, then G is at most $(2 + \lfloor 2\sqrt{\gamma} \rfloor)$-connected, by Theorem 4 of Cook (1973), and hence is not $(2 + \lfloor 2\sqrt{\gamma} \rfloor)$-extendable by Theorem 2.8. But since $2 + \lfloor 2\sqrt{\gamma} \rfloor \leq \left\lfloor \frac{9}{2} + \frac{18(\gamma-1)}{7+\sqrt{48\gamma-47}} \right\rfloor$, for all $\gamma \geq 1$, G is not $\left\lfloor \frac{9}{2} + \frac{18(\gamma-1)}{7+\sqrt{48\gamma-47}} \right\rfloor$ - extendable by Theorem 2.7.

So suppose G contains a triangle. (Note that this triangle need not be the boundary of a face in general.) Suppose also that G is $\left(\left\lfloor \frac{9}{2} + \frac{18(\gamma-1)}{7+\sqrt{48\gamma-47}} \right\rfloor\right)$- extendable. Then by Theorem 2.7, G must be 2-extendable. But then by Theorem 4.2 of Plummer (1980), G must be either bipartite or bicritical. But if G is bipartite, it contains no triangles, so it must be that G is bicritical.

From Lemma 2.3 we know that there exists a control point $v \in V(G)$ and from Lemma 2.5 for any such control point v, $\deg v \leq 6 + 12(\gamma-1)/p$. Using Lemma 2.6, we obtain

$$\begin{aligned} \deg v &\leq 6 + 12(\gamma-1)/p \\ &\leq 6 + \frac{12(\gamma-1)\cdot 2}{7+\sqrt{48\gamma-47}} \\ &= 6 + \frac{24(\gamma-1)}{7+\sqrt{48\gamma-47}}. \end{aligned} \tag{1}$$

(Note that the second inequality in the above series of three is strict whenever $\gamma > 1$.)

Let us say that a matching M in G **isolates** a point v if the lines of M cover $N(v)$, the set of neighbors of v, but M does not cover v. We will now show that given any control point v in G, there must be a matching in G which isolates v.

If there exists a point $w \notin \{v\} \cup N(v)$, then $G - v - w$ has a p.m. which certainly isolates v. So let us assume that $\{v\} \cup N(v) = V(G)$. Then by Lemma 2.5, we have $p = \deg v + 1 \leq 7 + (12(\gamma - 1))/p$. From this inequality it follows that $(p-2)/2 \leq (3 + \sqrt{1+48\gamma})/4$. Now by definition of n-extendable, $n \leq (p-2)/2$ and hence

$$\left\lfloor \frac{9}{2} + \frac{18(\gamma-1)}{7+\sqrt{48\gamma-47}} \right\rfloor \leq \left\lfloor \frac{3+\sqrt{1+48\gamma}}{4} \right\rfloor.$$

If we write this inequality as $\lfloor F(\gamma) \rfloor \leq \lfloor G(\gamma) \rfloor$, we claim that $F(\gamma) - 1 > G(\gamma)$ (the details are left to the reader) and hence we have a contradiction.

So we may assume that G contains a matching which isolates control point v.

Claim 1. If $\deg v = y$ and there are x triangular faces at point v, then if M_v is a *smallest* matching which isolates v, then $|M_v| \leq y - x/4$.

Proof of Claim 1. If $x = 0$, the inequality is trivially true, so suppose that $x > 0$. Since M_v is made up of two types of lines: (a) lines matching a point of $N(v)$ to a point of $V(G) - (N(v) \cup \{v\})$, and (b) lines matching two points of $N(v)$, we can obtain an upper bound for M_v once we have an upper bound for the total number of lines of type (a). When we have a matching at hand which matches a point u of $N(v)$ to a point not in $N(v) \cup \{v\}$, we will say that u is matched *out of* $N(v)$.

Note that no two points u_i, u_{i+1} in $N(v)$ which lie on the same triangular face at v can both be matched out of $N(v)$, for we could replace these two lines out of $N(v)$ with the line $u_i u_{i+1}$ and get a new matching which still isolates v, but is smaller than M_v, a contradiction of the choice of M_v. So we want an upper bound on the largest number of *potential* matching lines out of $N(v)$ subject to the constraint that no triangular face $v u_i u_{i+1} v$ has both u_i and u_{i+1} matched out of $N(v)$.

Without loss of generality, we assume that the various clusters of triangular faces are consecutive in, say, a cyclic clockwise array about point v; that is, two clusters are separated by exactly one non-triangular face. (See Figure 2.1.)

We now claim that the largest number of potential lines out of $N(v)$ from triangular faces in general is no larger than the number of such potential lines when all x triangles are in *one* triangle cluster.

FIGURE 2.1.

FIGURE 2.2.

FIGURE 2.3.

This can perhaps be most easily seen by an inductive procedure whereby we reduce the number of triangular clusters by one, but the number of potential matching lines out of $N(v)$ is never reduced. Let us call a cluster of triangular faces **odd** if the number of triangular faces in the cluster is odd and otherwise, **even**.

There are three cases to treat: (a) one odd and one even cluster, (b) two even clusters and (c) two odd clusters. See Figure 2.2 in which potential lines out of $N(v)$ belonging to an isolating matching are shown as arrows. In each case the dashed arrow represents the new potential line out of $N(v)$ obtained by the departure of a line lost in coalescing the two clusters. (The degree of v must remain constant of course.)

Note that in cases (a) and (b) above, the potential number of lines out of $N(v)$ stays the same under the indicated transformation while in

(c) it increases by one.

So now we shall suppose that the x triangular faces at v are consecutive. Let the points of the triangular faces be $u_0, \ldots, u_x$ (in this same clockwise order).

Recall that if any two consecutive u_i's are joined by a line of $E(G)$, but each is matched out of $N(v)$, then M_v is not a *smallest* isolating matching.

Let us first assume that $x < y$.

If x is odd, the largest possible number of M_v lines out of $u_0, \ldots, u_x$ is $(x+1)/2$. (See Figure 2.3.) If x is even, on the other hand, this largest possible number of M_v lines out of $u_0, \ldots, u_x$ is $(x+2)/2$. (See Figure 2.4.)

Now let us again suppose x is odd. In this case, there remain $x + 1 - ((x+1)/2) = (x+1)/2$ points of the x triangles which must be matched to other points in $N(v)$. Clearly, there will remain the most points of $N(v)$ available to be matched out of $N(v)$ when these $(x+1)/2$ points are perfectly matched with each other (when $(x+1)/2$ is even) and near-perfectly matched — with one matched to a neighbor of v not on a triangular face at v — in the case when $(x+1)/2$ is odd. So the maximum number of neighbors of v matched out of $N(v)$ is $= (x+1)/2 + y - (x+1)$ when $(x+1)/2$ is even and is $= (x+1)/2 + y - (x+1) - 1$ when $(x+1)/2$ is odd.

Thus when $(x+1)/2$ is odd, we have

$$|M_v| \leq \frac{(x+1)}{2} + (y - (x+2)) + \frac{(x+1)/2+1}{2} = y - \frac{(x+3)}{4},$$

while when $(x+1)/2$ is even, we have

$$|M_v| \leq \frac{(x+1)}{2} + (y - (x+1)) + \frac{1}{2}\frac{(x+1)}{2} = y - \frac{(x+1)}{4}.$$

Now suppose x is even. (See Figure 2.4.) Then there remain $(x + 1) - (x+2)/2 = x/2$ points of the x triangles which must be matched to other points in $N(v)$. As before, clearly there will remain the most points of $N(v)$ available to be matched out of $N(v)$ when these $x/2$ points are perfectly matched pairwise when $x/2$ is even and near-perfectly matched pairwise — with one matched to a neighbor of v not on a triangular face at v — when $x/2$ is odd.

So if $x/2$ is odd, we have

$$|M_v| \leq \frac{(x+2)}{2} + y - (x+2) + \frac{1}{2}(x/2+1) = y - \frac{(x+2)}{4},$$

while if $x/2$ is even, we have

$$|M_v| \leq \frac{(x+2)}{2} + y - (x+1) + \frac{1}{2}(x/2) = y - x/4.$$

Now let us assume that $x = y$; in other words, that all faces at y are triangles. Label $N(v)$ clockwise as $u_1, \ldots, u_x$. First suppose that x is even. Then $\{u_1u_2, u_3u_4, \ldots, u_{x-1}u_x\}$ is a matching isolating v and hence $|M_v| = x/2 = y - x/2 < y - x/4$.

Finally, suppose that x is odd. If any of the u_i's can be matched out of $N(v) \cup \{v\}$, say u_1 for example is matched to a point w, then $\{u_1w, u_2u_3, \ldots, u_{x-1}u_x\}$ is a matching isolating v and so $|M_v| = (x-1)/2 + 1 = (x+1)/2 = x - (x-1)/2 = y - (x-1)/2 \leq y - x/4$, since $x = y \geq 5$. (Recall that by Theorem 2.7, graph G is 4-extendable and hence by Theorem 2.8, G is 5-connected. Hence $\operatorname{mindeg} G \geq 5$.)

So assume that $\{v, u_1, \ldots, u_x\} = V(G)$. But then G has no matching which isolates v, a contradiction.

So we see that in all cases, $|M_v| \leq y - x/4$. This completes the proof of Claim 1.

So let $f(x,y) = y - (x/4)$. We would now like to find an upper bound for $f(x,y)$ in terms of γ. From Corollary 2.4 and Lemma 2.6 we have

$$\frac{y}{2} \leq \sum_{i=1}^{y} \frac{1}{x_i} + 1 + \frac{2(\gamma-1)}{p}$$

$$< \sum_{i=1}^{y} \frac{1}{x_i} + 1 + \frac{4(\gamma-1)}{7+\sqrt{48\gamma-47}}$$

$$\leq \frac{x}{3} + \frac{y-x}{4} + \frac{4(\gamma-1)}{7+\sqrt{48\gamma-47}} + 1.$$

Hence

$$y < \frac{x}{3} + \frac{16(\gamma-1)}{7+\sqrt{48\gamma-47}} + 4. \qquad (2)$$

Thus we have the linear program:

$$\begin{array}{lll} \text{maximize} & f(x,y) = y - (x/4) & \\ \text{subject to} & y \leq 6 + \frac{24(\gamma-1)}{7+\sqrt{48\gamma-47}} & (1) \\ & y < \frac{x}{3} + 4 + \frac{16(\gamma-1)}{7+\sqrt{48\gamma-47}} & (2) \\ & 0 \leq x \leq y & (3) \\ & 5 \leq y. & (4) \end{array}$$

The feasible region is a quadrilateral having vertices $A = (0, 4 + 16r(\gamma))$, $B = (6 + 24r(\gamma), 6 + 24r(\gamma))$, $C = (5,5)$ and $D = (0,5)$, where $r(\gamma) = \frac{\gamma-1}{7+\sqrt{48\gamma-47}}$. Hence the maximum value for $f(x,y)$ occurs at vertex B and this value is $\frac{9}{2} + \frac{18(\gamma-1)}{7+\sqrt{48\gamma-47}}$.

Now since we may select only integral numbers of lines, we can isolate v by choosing at most $\left\lfloor \frac{9}{2} + \frac{18(\gamma-1)}{7+\sqrt{48\gamma-47}} \right\rfloor$ lines which collectively cover $N(v)$ and the proof is complete. ■

FIGURE 3.1.

3. The special case when $\gamma = 1$

When G is "toroidal", that is, when $\gamma(G) = 1$, we can obtain a sharper result than that implied by Theorem 2.9.

3.1. THEOREM. *If G is connected and $\gamma(G) = 1$, then either*

(a) G is not 3-extendable or

(b) G is point-regular of degree 4 and face-regular of degree 4 and hence not 4-extendable.

PROOF. From Lemmas 2.1 and 2.5 we know G contains a point v with $\Phi(v) \geq 0$ and $\deg v \leq 6$. Now suppose G is 3-extendable.

Case I. Suppose $\Phi(v) > 0$, for some point v. (As before, we will call such a point v a *control point*.) Then $\deg v \leq 5$ by Lemma 2.2. Now by definition of n-extendability, $p \geq 8$. Also we know that G is 4-connected by Theorem 2.8 and hence $\deg v \geq 4$.

First let us assume that $\deg v = 4$. Now the solutions to the diophantine inequality

$$\sum_{i=1}^{4} \frac{1}{x_i} > 1$$

are as listed in Plummer (1985) and we proceed to treat each.

$(3,3,3,x)$, $x \geq 3$. Let the neighbors of v be $\{a,b,c,d\}$. Then note that no two of these four neighbors of v can be equal since G has no multiple lines. Thus $\{ab, cd\}$ do not extend to a p.m. and hence G is not 2-extendable. But then by Theorem 2.7, G is not 3-extendable either, a contradiction. (See Figure 3.1.)

$(3,3,4,x)$, $4 \leq x \leq 11$. Let the fourth point of the 4-face at v be e.

(i) Suppose the two triangular faces are consecutive in, say, the clockwise direction about point v. We know that $e \neq c, d$ or b, again since there are no multiple lines. So suppose $e = a$. Then $\{ad, bc\}$ do not extend and we have a contradiction as before. (See Figure 3.2(i).)

So $e \cap \{a,b,c,d\} = \emptyset$. Now if there exists a point $g \notin \{a,b,c,d,e\}$, but which is adjacent to one of the points a, c, d, we get a matching of

FIGURE 3.2.

FIGURE 3.3.

size 3 at v which does not extend, again a contradiction. Thus since G has at least 8 points, we have that $\{b, e\}$ is a cutset of G and hence G is not 3-connected and hence not 2-extendable, a contradiction. (See Figure 3.2(ii).)

(ii) Now suppose the two triangular faces are separated by the 4-gonal face. Then $\{ab, cd\}$ is a matching of size 2 which does not extend, a contradiction.

(3,3,5,x), $x = 5, 6, 7$. There are two cases to consider.

First let us suppose that the two triangular faces at v are consecutive in the clockwise orientation about point v. Let d be the point on the pentagonal face at v which is adjacent to point c via a line of the pentagonal face boundary. Also let the remaining fourth point adjacent with point v be denoted e.

Now we know that d cannot be equal to b, c or e, so suppose that $d = a$. (See Figure 3.3(i).)

Now if points e and b are adjacent then $\{be, ac\}$ does not extend, a contradiction. So suppose that e and b are not adjacent.

Now if e and a are adjacent, then matching $\{ea, bc\}$ does not extend, while, symmetrically, if e and c are adjacent then matching $\{ec, ab\}$ does not extend. In either case we have a contradiction.

So point e is adjacent to a point g, $g \notin \{a = d, b, c\}$ and we may assume that g is such that line eg is on the boundary of the pentagonal face at v. Thus we may assume that g and $a = d$ are adjacent on the pentagonal face.

Now since $\deg b \geq 4$, we must have a point h adjacent to b such that $h \notin \{a = d, b, c, v\}$. We already know that $h \neq e$ from above. If

FIGURE 3.4.

FIGURE 3.5.

$h \neq g$, then matching $\{bh, ac, eg\}$ does not extend and once again we have a contradiction.

So we must assume that $g = h$. (See Figure 3.3(ii).) Now we already know that e is not adjacent to any of $a = d, b, c$. Since $\deg e \geq 4$, there must be yet another point j, where $j \notin \{a = d, b, c, v, g = h\}$. See Figure 3.3(iii). But the matching $\{ej, ag, bc\}$ does not extend, a contradiction.

So now we may suppose that $d \neq a$. Hence $d \notin \{a, b, c, v, e\}$. Now let f be the fifth point of the pentagonal face at v. So the pentagonal face boundary at v is $vcdfev$. If $f \notin \{a, b, c\}$, then the matching $\{ab, cd, ef\}$ does not extend, a contradiction. (See Figure 3.4(i).)

So we assume that $f \in \{a, b, c\}$. But $f \neq c$ since G contains no multiple lines. So $f \in \{a, b\}$.

First suppose that $f = a$. (See Figure 3.4(ii).) In this case, $\{ae, bc\}$ does not extend.

Now suppose that $f = b$. (See Figure 3.5.)

If a is adjacent to some point g, where $g \notin \{b = f, c, d, v, e\}$, then matching $\{ag, be, cd\}$ does not extend, a contradiction. Therefore, since $\deg a \geq 4$, a must be adjacent to some two points in the set $\{c, d, e\}$. If a is adjacent to c, the matching $\{ac, be\}$ does not extend, a contradiction. So a is adjacent to both d and e. But then matching $\{ae, bc\}$ does not extend, a contradiction.

This finishes the case when the two triangular faces at v are consecutive.

Now assume that the two triangular faces at v are separated by the pentagonal face. (See Figure 3.6.) But then matching $\{ab, cd\}$ does not extend. This completes the case for $(3, 3, 5, x)$, $x = 5, 6, 7$.

FIGURE 3.6.

FIGURE 3.7.

FIGURE 3.8.

$(3,4,4,x)$, $x = 4,5$. First suppose that the two quadrilateral faces at v are not separated by the triangular face. Without loss of generality, suppose the neighbors of v in a clockwise orientation about v are a, b, c, d respectively, where $abva$ is the triangular face. Let $vbecv$ be the 4-gonal face contiguous to the triangular face, moving clockwise.

Suppose now that $e = a$. (See Figure 3.7.) In this case, since G has no parallel lines, $\deg b = 2$, a contradiction.

So suppose $a \neq e$. If $e = d$, then matching $\{ab, cd\}$ does not extend. So we may suppose $e \neq a, d$. (See Figure 3.8.)

First note that c is not adjacent to d or we would have a matching of size 2 which does not extend. Then we must have $N(d) \subseteq \{a, b, e, v\}$ or else we would have a matching of size 3 which does not extend. But then since $\deg d \geq 4$, we must have $N(d) = \{a, b, e, v\}$. Next consider $N(a)$. Point a is not adjacent to c or else we would have a matching of size 2 which does not extend. So we must have $N(a) \subseteq \{b, e, d, v\}$ or else as above we would have a matching of size 3 which does not extend. So since $\deg a \geq 4$ also, we get that $N(a) = \{b, d, e, v\}$. (See Figure 3.9.)

Applying the same arguments to possible neighbors of point b, we see that $N(b) = \{a, d, e, v\}$ too and hence, since $|(VG)| \geq 8$, since G is

FIGURE 3.9.

FIGURE 3.10.

3-extendable, we must have that $\{c, e\}$ is a cutset for G, contradicting the fact that, by Theorem 2.8, G must be 4-connected.

Now we turn to the case in which the two quadrilateral faces at v are separated in the clockwise order about v by a triangular face. Let the first quadrilateral face be denoted $vaebv$, the contiguous triangular face by $vbcv$, and the next contiguous quadrilateral face by $vcfdv$. First of all, note that points a and d are not adjacent or we would have a matching of size 2 which does not extend. Thus $d \neq e$.

Suppose next that $e = c$. Then since G has no parallel lines, $\deg b = 2$, a contradiction.

Thus $e \notin \{a, b, c, d, v\}$. By symmetry, $f \notin \{a, b, c, d, v\}$ as well.

Suppose now that $e = f$. (See Figure 3.10.)

Now $N(d) \subseteq \{e = f, b, c, v\}$, or else we get a matching containing ae and bc, and covering d, which does not extend. By 4-connectivity, then, $N(d) = \{e = f, b, c, v\}$.

Similarly, $N(a) \subseteq \{e = f, b, c, d, v\}$. But $b \notin N(a)$, or else $\{ab, cd\}$ does not extend. We already know that $d \notin N(a)$, so $N(a) \subseteq \{e = f, c, v\}$, contradicting the 4-connectivity of G.

Thus we may assume that $e \neq f$. (See Figure 3.11.) But then $\{ae, bc, fd\}$ does not extend; a contradiction. The case $(3, 4, 4, x)$ is thus complete.

Now suppose $\deg v = 5$. Then we have the inequality

$$\sum_{i=1}^{5} \frac{1}{x_i} > \frac{3}{2}.$$

The only solutions in this case are:

FIGURE 3.11.

$(3,3,3,3,x)$, $x = 3,4,5$. Let $N(v) = \{a,b,c,d,e\}$. Now if there is a point $g \notin N(v)$ such that either a is adjacent to g or e is adjacent to g, we are done since the matching $\{ag, bc, de\}$ (respectively, $\{ab, cd, eg\}$) does not extend, a contradiction.

We get a similar contradiction if there is a point $g \notin N(v)$ such that g is adjacent to point c. But then since G must have at least 8 points, since we are assuming it is 3-extendable, it follows that $\{b,d\}$ is a cutset for G and hence G is not 3-connected, a contradiction.

Case II. So we may assume that $\Phi(v) \leq 0$ for all $v \in V(G)$. But then by Lemma 2.1, we have $\Phi(v) = 0$ for all $v \in V(G)$. Choose an arbitrary point v in $V(G)$. Then by Lemma 2.2, $0 = \Phi(v) \leq 1 - \deg v/6$, so $\deg v \leq 6$.

We also now have $\Phi(v) = 1 - \deg v/2 + \sum_{i=1}^{\deg v} 1/x_i = 0$, or $\sum_{i=1}^{\deg v} 1/x_i = \deg v/2 - 1$. Moreover, since G is 3-extendable, it is 4-connected and hence $\deg v \geq 4$.

First suppose $\deg v = 4$ and hence $\sum_{i=1}^{4} 1/x_i = 1$. Let $N(v) = \{a,b,c,d\}$ in the usual clockwise orientation about v. Also let the faces with face angles at v be named F_1, F_2, F_3, F_4 under the same orientation where face F_1 contains the lines av and vb.

There are only four solutions to this diophantine equation: $(3,3,4,12), (3,3,6,6), (3,4,4,6)$, and $(4,4,4,4)$. Arguments identical to those in Case I give us contradictions for the solutions $(3,3,4,12)$ and $(3,4,4,6)$.

Let us consider the case $(3,3,6,6)$. If the two triangular faces are not consecutive in the clockwise orientation about v, then we have a matching of size 2 which does not extend and again we have a contradiction. So we may assume that the two triangular faces at v are consecutive in the clockwise ordering. Let $N(v) = \{a,b,c,d\}$ where $abva$ and $bcvb$ are these triangular faces. Note that a and d are not adjacent, or else $\{bc, ad\}$ would not extend. Similarly, points c and d are not adjacent.

Since $\deg c \geq 4$, point c is adjacent to some point f where $f \notin \{a,b,d,v\}$. Similarly, point d is adjacent to at least two points not in $\{a,b,c,v\}$ and hence to a point $g \notin \{a,b,c,v,f\}$. But then $\{ab, cf, dg\}$

is a matching of size 3 which does not extend.

We defer the remaining case $(4,4,4,4)$ until the end of this proof.

Next, suppose $\deg v = 5$ and let $N(v) = \{a,b,c,d,e\}$. Then $\sum_{i=1}^{5} 1/x_i = 3/2$. It is then clear that $x_i = 3$ for at least two of the x_i's, but for no more than four.

Suppose first that more than two x_i's $= 3$. In fact, suppose first that four of the faces at v are triangles and hence the fifth must be a hexagon.

Again, since G is 3-extendable, $|V(G)| \geq 8$ and so there must be a line with exactly one endpoint in the set $N(v)$. If such a line is incident with a, c, or e, we get a contradiction as before. So such a point can only be incident with b or d. But then $\{b,d\}$ must be a cutset of G, a contradiction.

Suppose next that there are precisely three triangular faces at v and hence the two remaining faces must both be quadrilaterals. There are two subcases to consider here. First suppose that all three triangular faces are consecutive in a clockwise orientation about point v. Without loss of generality, suppose that faces $avba$, $bvcb$ and $cvdc$ are these triangular faces. Now as before, if point e is adjacent to a point different from a,b,c,d or v, then G is not 3-extendable. So $N(e) \subseteq \{a,b,c,d,v\}$. Since $\deg e \geq 4$, we may assume that e is adjacent to b without loss of generality. Now if a were adjacent to a point g not in $\{b,c,d,e,v\}$ then again G is not 3-extendable. Thus we may assume that $N(a) \subseteq \{b,c,d,e,v\}$. But then again since $|V(G)| \geq 8$, $\{b,c,d\}$ is a cutset contradicting the fact that G is 4-connected.

So now suppose that the three triangular faces at v are not consecutive about v. Without loss of generality, assume that $abva$, $bcvb$ and $devd$ are the triangular faces at v. Now if either a or c is adjacent to a point not in the set $\{a,b,c,d,e,v\}$, we find that G is not 3-extendable. Thus $N(a) \subseteq \{b,c,d,e,v\}$ and similarly for $N(c)$. Thus $\{b,d,e\}$ is a cutset of G, once again contradicting the fact that G is 4-connected.

So suppose that exactly two x_i's $= 3$, say x_1 and x_2. Then $1/x_3 + 1/x_4 + 1/x_5 = 3/2 - 2/3 = 5/6$, and this contradicts the fact that $1/x_3 + 1/x_4 + 1/x_5 \leq 3/4$.

Finally, suppose $\deg v = 6$ and hence $\sum_{i=1}^{6} 1/x_i = 2$. Then the only solution to this diophantine equation is $(3,3,3,3,3,3)$ but since $|V(G)| \geq 8$, it is immediate that G is not 3-extendable.

It remains only to consider the solution $(4,4,4,4)$. Moreover, since all other cases have been treated, we may assume that G is both 4-point-regular and 4-face-regular. But such a graph can be at most 4-connected

FIGURE 3.12.

and hence not 4-extendable. ■

An infinite family of graphs of genus 1 which are 3-extendable is the family of **bipartite toroidal lattices** $T(2m, 2n)$, for $m, n \geq 2$, where $T(2m, 2n)$ is just the Cartesian product of the even cycles C_{2m} and C_{2n} on $2m$ and $2n$ points respectively. Clearly these graphs can be imbedded on the torus in such a way that each face is a quadrilateral. We show $T(4, 6)$ so imbedded in Figure 3.12. (Note that $T(4, 4)$ is perhaps better known as the 4-cube, Q_4.)

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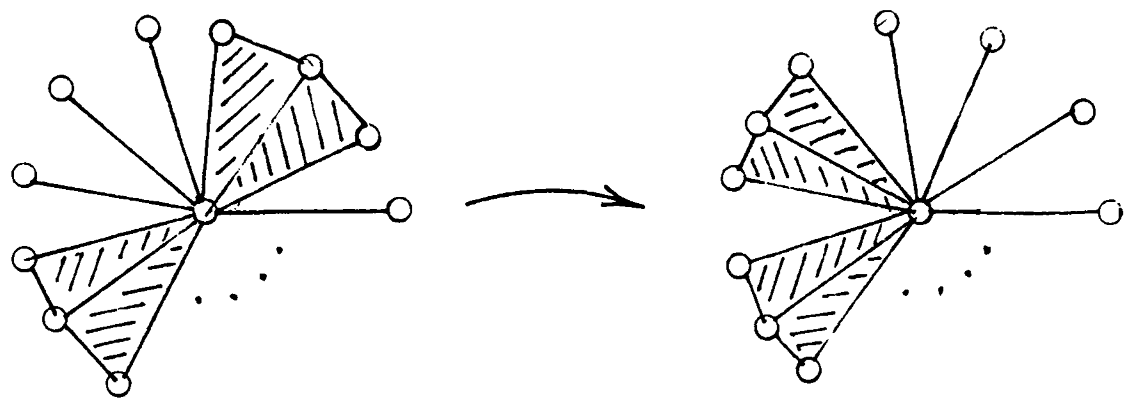
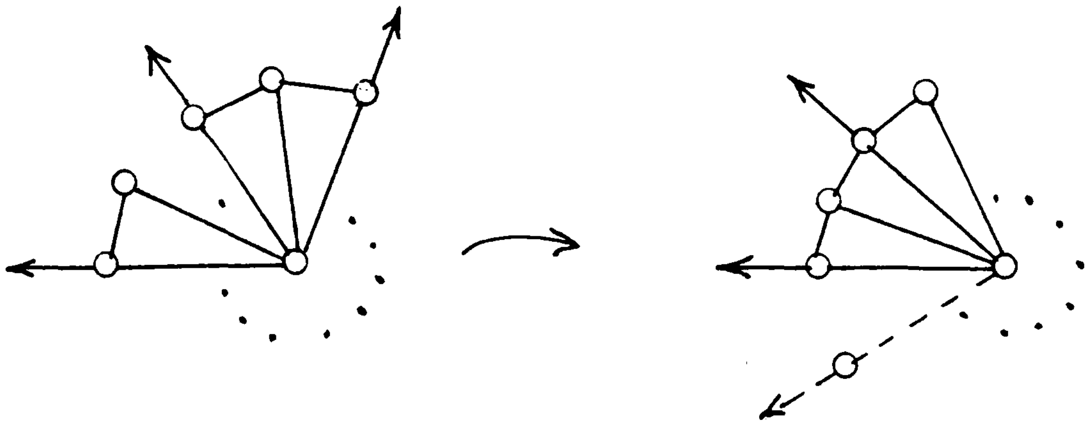
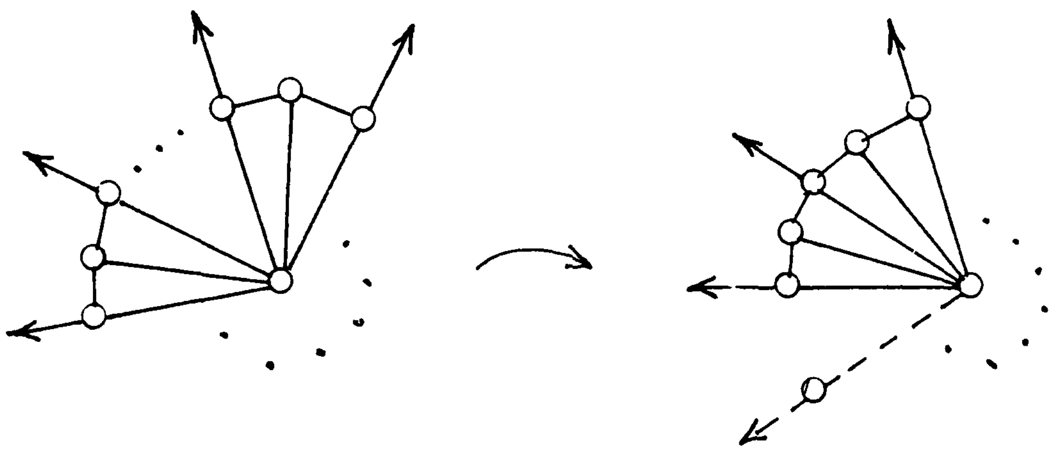


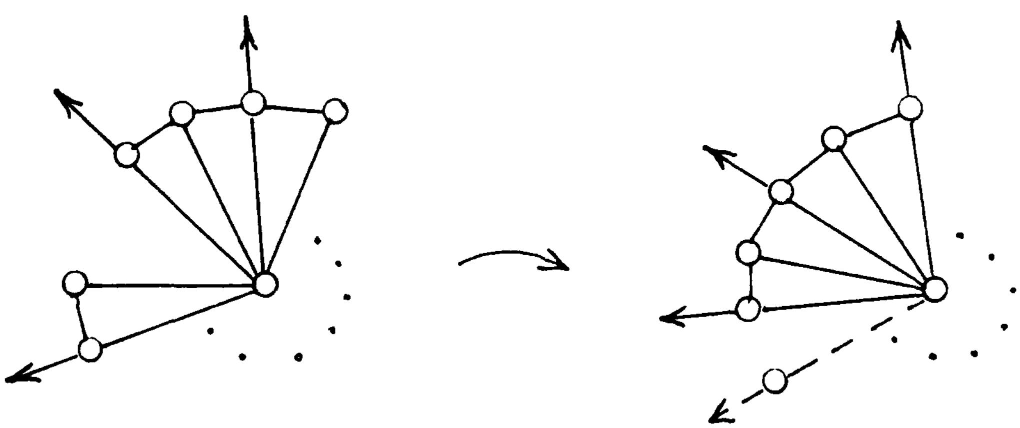
Figure 2.1



(a) One odd and one even cluster

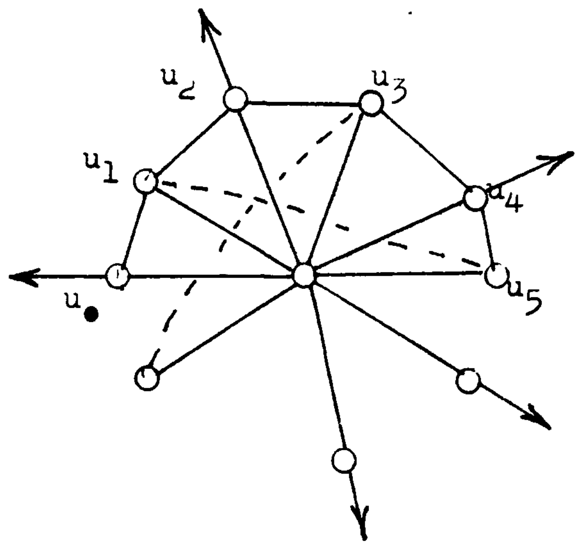


(b) Two even clusters

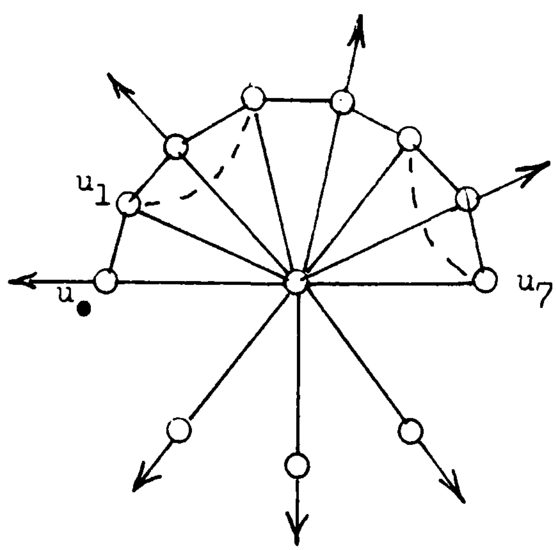


(c) Two odd clusters

Figure 2.2

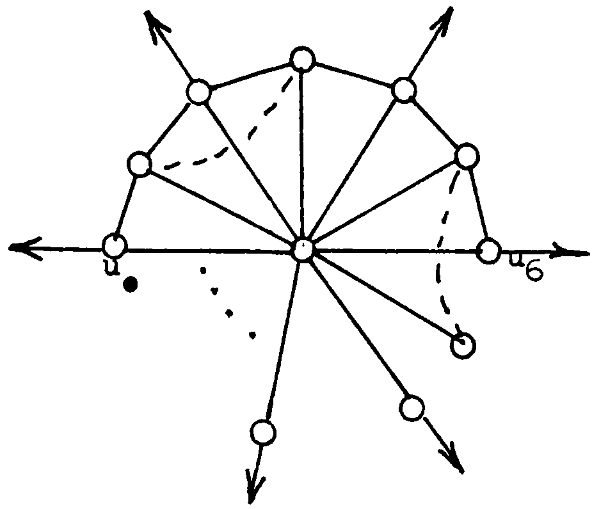


(a) x odd and $\frac{x+1}{2}$ odd

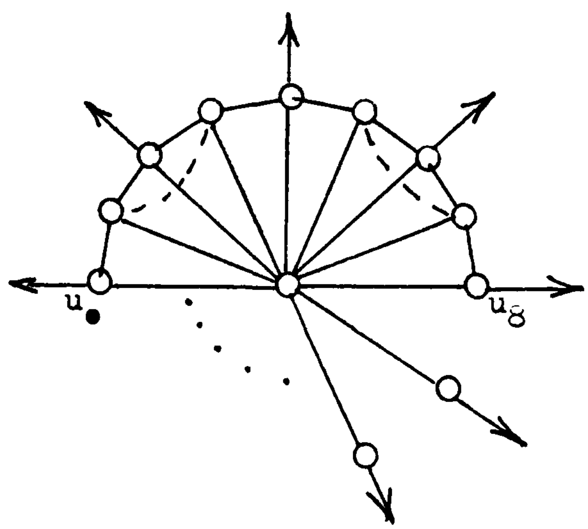


(b) x odd and $\frac{x+1}{2}$ even

Figure 2.3



(a) x even and $\frac{x}{2}$ odd



(b) x even and $\frac{x}{2}$ even

Figure 2.4

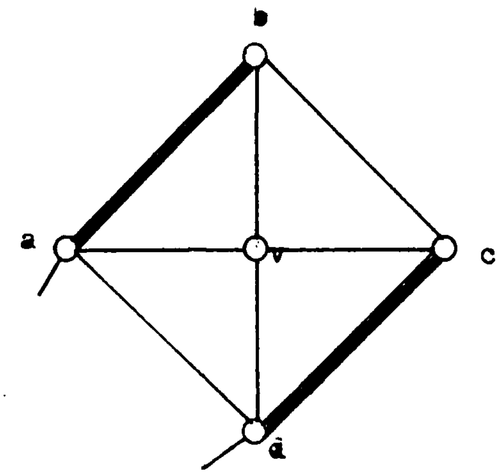
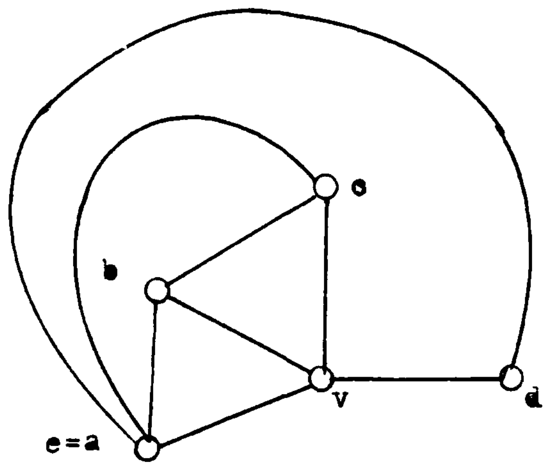


Figure 3.1



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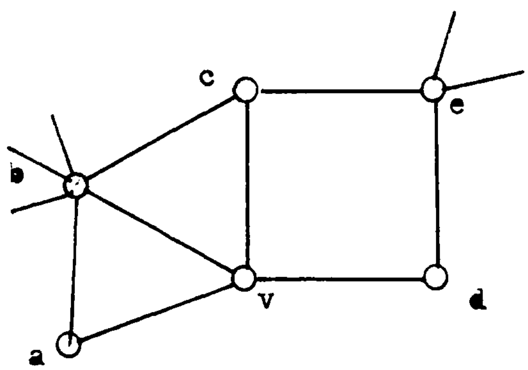
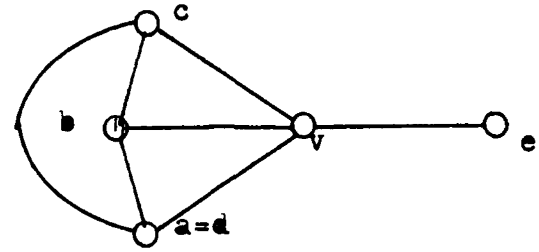


(ii)

Figure 3.2



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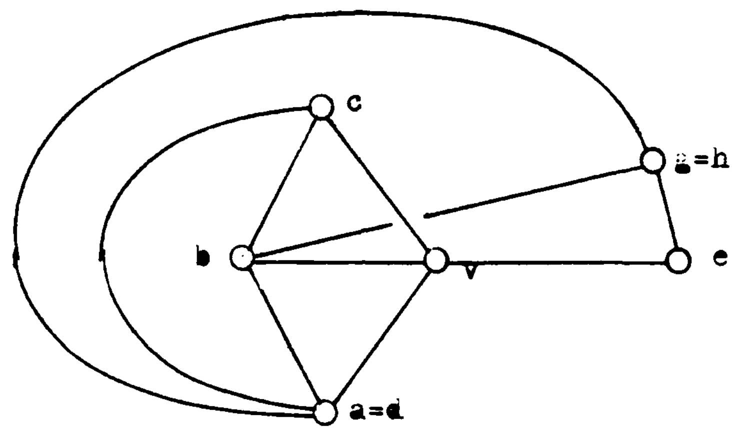

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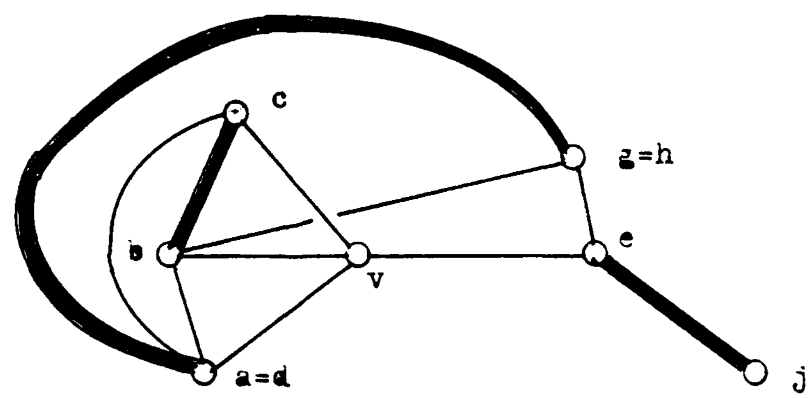
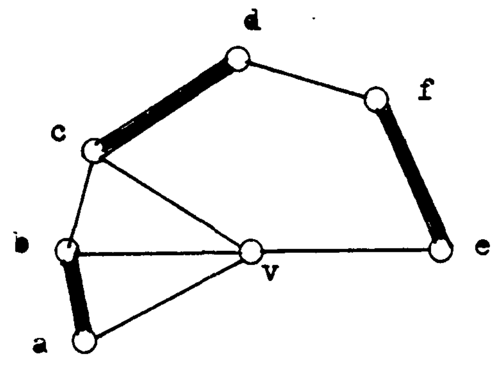
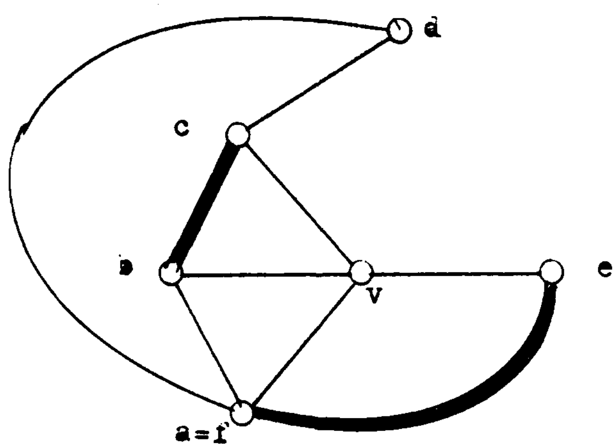


(iii)

Figure 3.3

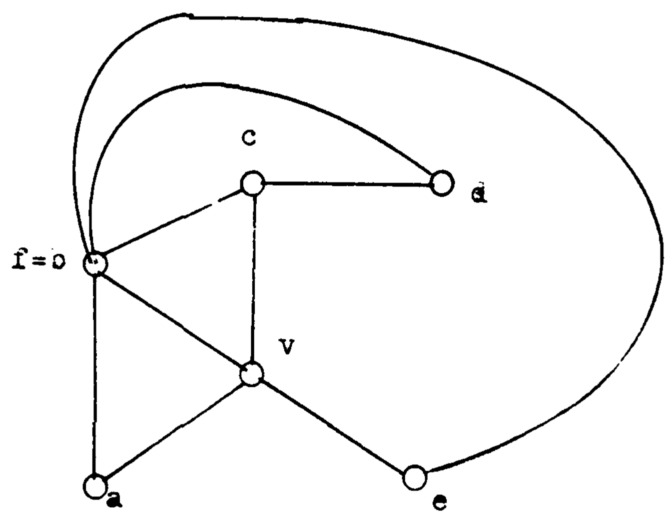


(i)



(ii)

Figure 3.4



Fi.ure 3.5

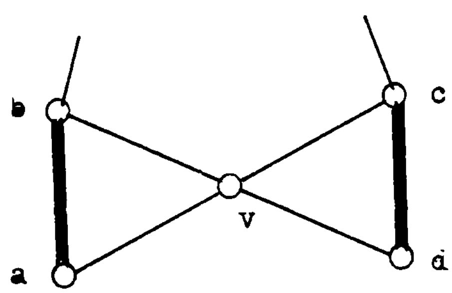


Figure 3.6

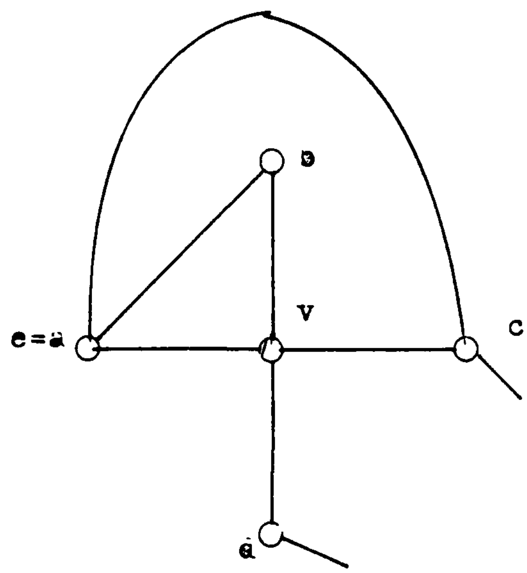


Figure 3.7

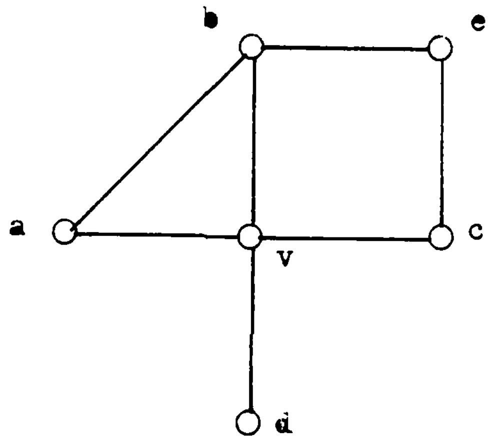


Figure 3.8

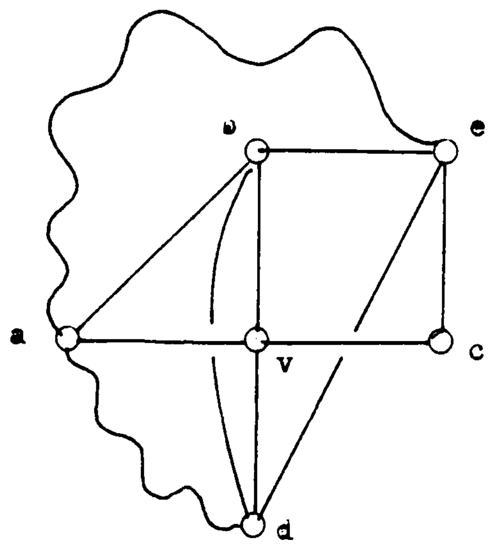


Figure 3.9

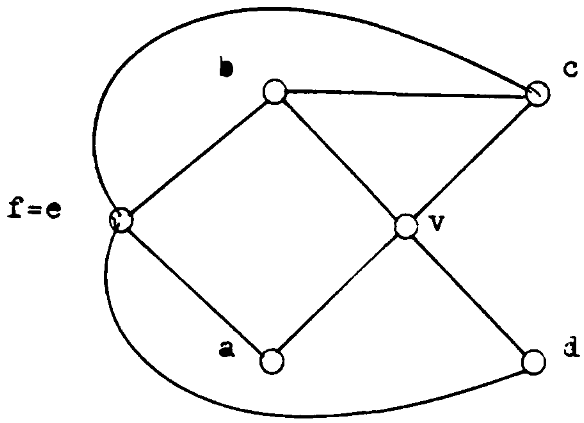


Figure 3.10

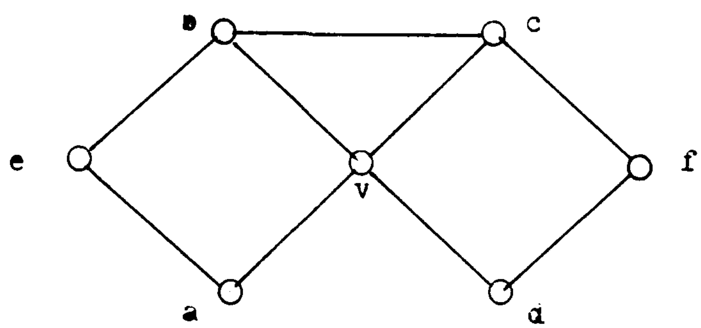


Figure 3.11

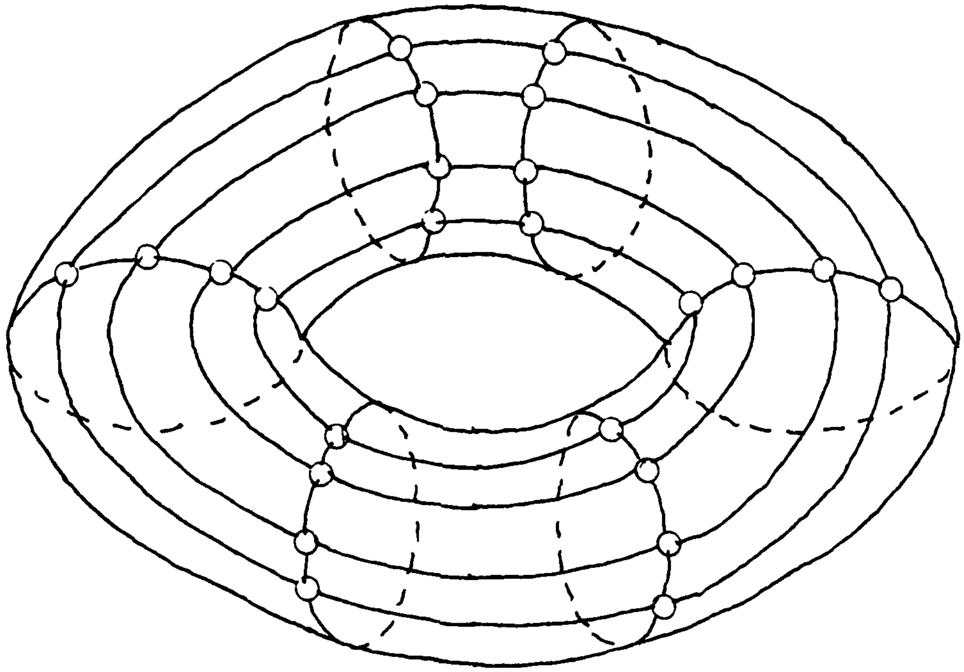


Figure 5.12

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